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A Factorization of Determinant Related to Some Random Matrices

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We consider the expectation of the determinant $\det(\lambda - X)^{-1}$ for $\operatorname{Im} \lambda > 0$ associated with some random $N \times N$ matrices and factorize it into N Stieltjes transforms of probability measures. Moreover, using this factorization, we investigate the limiting behavior of the logarithm of the quantity as $N \to \infty$.

KEY WORDS: Scattering problem; random matrix (GUE); factorization; orthogonal polynomials; semicircle law.

1. INTRODUCTION

Let X be an $N \times N$ Hermitian matrix, then it is elementary fact that

$$\det(\lambda - X)^{-1} = \prod_{i=1}^{N} (\lambda - x_i)^{-1}$$

where $\{x_i\}_{i=1}^{N}$ are real eigenvalues of X. If $X = X(\omega)$ is a random Hermitian matrix on a probability space (Ω, P) in some sense, for each ω, of course, the equation above also holds. Can we make sense of it after taking expectation? In other words, do there exist probability measures $\{\mu_i(dx)\}_{i=1}^{N}$ on $\mathbf{R}$ satisfying

$$E[\det(\lambda - X)^{-1}] = \prod_{i=1}^{N} \int_{\mathbf{R}} (\lambda - x)^{-1} \mu_i(dx) \tag{1.1}$$

where E is the expectation with respect to P. This problem is considered as an example in order to investigate some quantities related to scattering problems for discrete Schrödinger operators.

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In ref. 3 they used the so-called Krein's spectral shift function defined by

$$\zeta(\lambda, a) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \log g_{\lambda + i\varepsilon}(a, a) \tag{1.2}$$

where $g_\lambda(a, a)$ is the green function (or the resolvent kernel of $\mathscr{L} = -\Delta + V$). This is an important quantity in the scattering theory and using it they showed several trace formulas for one-dimensional Schrödinger operators systematically.

The author dealt with general graphs in place of the real line $\mathbf{R}$ and showed two types of trace formulas.[7] In this case, the quantity $\operatorname{Im} \log \det G_\lambda$ is used instead of $\operatorname{Im} \log g_\lambda(a, a)$, where G_λ is a finite matrix whose elements are the green functions. It is important to know the properties of $\det G_\lambda$ in our setting.

Let G be an infinite graph and Δ the discrete Laplacian on $\ell^2(G)$ which is defined by $\Delta = P - I$, where P is a transition operator. Let V be a real-valued bounded function on G and $\mathscr{L} = -\Delta + V$. Using the green function $g_\lambda(x, y)$ of $\mathscr{L}$, for a finite subgraph A of G, we define a $|A| \times |A|$ finite matrix $G_\lambda^A = (g_\lambda(a, b))_{a, b \in A}$. Then we can show the following:

Proposition 1.1. Let G_λ^A be the matrix defined as above and let $\sigma(\mathscr{L})$ be the spectrum of the discrete Schrödinger operator $\mathscr{L}$. Let N be the cardinality of A. Then $\det G_\lambda^A$ is non-zero analytic on $\mathbf{C} \backslash [\lambda_0, \lambda_\infty]$, where $\lambda_0 = \inf \sigma(\mathscr{L})$ and $\lambda_\infty = \sup \sigma(\mathscr{L})$. Moreover, it has an integral representation, that is, there exists a positive probability measure on $\sigma(\mathscr{L})^N$ such that

$$\det G_\lambda^A = \int_{\sigma(\mathscr{L})^N} \prod_{i=1}^{N} (\lambda - x_i)^{-1} \, \nu(dx_1 \cdots dx_N) \tag{1.3}$$

Here we omit the proof, however, it is important to remark that from the way of the construction of the measure $\nu(dx_1 \cdots dx_N)$ is not a product measure.

We give an example which can be easily calculated.

Example 1.2. Let G be a d-regular tree and A an arbitrary connected finite subset of G with cardinality N. Let P be the transition operator associated with the simple random walk on G, $\Delta = I - P$ and $V \equiv 0$. In this case, $\mathscr{L} = -\Delta$. It is well known that $\sigma(-\Delta) = [1 - \alpha, 1 + \alpha]$ where $\alpha = \alpha_d = 2\sqrt{d-1}/d$. By Proposition 1.1, $\det G_\lambda^A$ is represented by the integral of the form (1.3). However, we can choose a product measure instead of ν in Proposition 1.1

$$\det G_\lambda^A = \int_{\sigma(-\Delta)^N} \prod_{i=1}^{N} \frac{1}{\lambda - x_i} \, m(dx_1) \otimes n(dx_2) \otimes \cdots \otimes n(dx_N) \tag{1.4}$$

where $m(dx) = (d/2\pi)(\sqrt{\alpha^2 - (x-1)^2}/(1-(x-1)^2))\,dx$, $n(dx) = (2/\pi) \times \sqrt{1 - \alpha^{-2}(x-1)^2}\,dx$.

This example shows the possibility of choosing a product probability measure without changing the integral. Moreover, from this example, we can conclude that

$$\lim_{|A| \to \infty} \frac{1}{|A|} \log \det G_\lambda^A = \log \int_{\mathbf{R}^1} \frac{1}{\lambda - x}\, n(dx) \tag{1.5}$$

This implies the semi-circle law.

In view of Proposition 1.1 and Example 1.2, it is natural to ask that for a probability measure ν on $\mathbf{R}^n$, do there exist probability measures $\{\mu_i(dx)\}_{i=1}^N$ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}^N} \prod_{i=1}^N (\lambda - x_i)^{-1}\, \nu(dx_1 \cdots dx_N) = \prod_{i=1}^N \int_{\mathbf{R}} (\lambda - x)^{-1}\, \mu_i(dx)$$

This means that the left-hand side is factorized into N Stieltjes transforms. The first question (1.1) can be considered as the same one as above since there exists a probability measure $\nu(dx_1 \cdots dx_N)$ on $\mathbf{R}^N$ (which is the joint distribution of N eigenvalues and not in general a product measure) such that

$$E[\det(\lambda - X)^{-1}] = \int_{\mathbf{R}^N} \prod_{i=1}^N (\lambda - x_i)^{-1}\, \nu(dx_1 \cdots dx_N)$$

In general, the answer of the question is a no. For example, we take a probability measure $1/2(\delta_{(1,1)} + \delta_{(-1,-1)})$ on $\mathbf{R}^2$ as ν where $\delta_{(a,b)}$ is a unit mass on $(a, b) \in \mathbf{R}^2$, and we have

$$\int_{\mathbf{R}^2} \prod_{i=1}^2 (\lambda - x_i)^{-1}\, \nu(dx_1\, dx_2) = \frac{\lambda^2 + 1}{(\lambda - 1)^2 (\lambda + 1)^2} \tag{1.6}$$

The right-hand side has a zero in the upper half plane. But, if the left-hand side could be factorized into two Stieltjes transforms, it cannot have any zero in the upper half plane, and it is a contradiction. However, we conjecture that the question above can be affirmatively solved for measures which come from determinants in some sense.

In this paper, as an example for the question above, we deal with a certain class of random matrices which is closely related to the Gaussian Unitary Ensemble, and we will show the possibility of a factorization and the semi-circle law for simple cases as an easy corollary to it.

Let $\mathscr{H}_N$ be the space of all $N \times N$ Hermitian matrices, i.e.,

$$\mathscr{H}_N = \{X \in M_N;\ X^* = X\}$$

where M_N is the totality of $N \times N$-matrices. We consider the following probability measure on $\mathscr{H}_N$:

$$P_N(dX) = Z_N^{-1} \exp(-\mathrm{Tr}\, V(X))\, dX$$

where V is a real-valued function, dX is the Lebesgue measure over N^2 independent elements of matrices, Z_N is a normalization constant (a partition function).

Now, $X \in \mathscr{H}_N$ can be diagonalized as

$$X = U^* D U$$

where D is a diagonal matrix with elements $\{x_1, x_2,..., x_N\}$. Noting that $\mathrm{Tr}\, V(X)$ depends only on the N eigenvalues of X, we integrate all variables except $\{x_1, x_2,..., x_N\}$. Then we obtain the joint eigenvalue distribution of $\{x_1, x_2,..., x_N\}$ as follows:

$$P_N(x_1, x_2,..., x_N) = C_N^{-1} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2 \exp\left(-\sum_{k=1}^{N} V(x_k)\right) dx_1\, dx_2 \cdots dx_N$$

where C_N is a normalization constant which is called Selberg integral.[8, 9]

Now we consider the following quantity

$$E_N[\det(\lambda - X)^{-1}] \qquad \mathrm{Im}\, \lambda > 0$$

where E_N is the expectation with respect to the probability measure P_N on $\mathscr{H}_N$. Then we have

$$E_N[\det(\lambda - X)^{-1}]$$
$$= C_N^{-1} \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left(\frac{1}{\lambda - x_i}\right) \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2 \exp\left(-\sum_{k=1}^{N} V(x_k)\right) dx_1 \cdots dx_N$$

We consider more general setting than above problem. Let $\mu(dx)$ be a probability measure on $\mathbf{R}$ with moments of all order and infinitely many points of increase. We define

$$I_N(\lambda) = C_N^{-1} \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left(\frac{1}{\lambda - x_i}\right) \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2\, \mu_N(dx_1\, dx_2 \cdots dx_N) \tag{1.7}$$

where $\mu_N(dx_1\, dx_2 \cdots dx_N) = \mu(dx_1)\, \mu(dx_2) \cdots \mu(dx_N)$ and

$$C_N = \int_{\mathbf{R}^N} \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)^2\, \mu_N(dx_1\, dx_2 \cdots dx_N)$$

Our theorem is the following:

Theorem 1.3. Let $p_n(x)$ be an orthonormal polynomial of degree n with respect to the measure $\mu(dx)$. Then

$$\begin{aligned} I_N(\lambda) &= \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left(\frac{1}{\lambda - x_i}\right) \delta_{\lambda_1}(dx_1) \otimes \cdots \otimes \delta_{\lambda_{N-1}}(dx_{N-1}) \otimes p_{N-1}^2(x_N)\, \mu(dx_N) \\ &= \frac{k_{N-1}}{p_{N-1}(\lambda)} \int_{\mathbf{R}^1} \frac{1}{\lambda - x}\, p_{N-1}(x)^2\, \mu(dx) \end{aligned} \tag{1.8}$$

where $\lambda_1,..., \lambda_{N-1}$ are the zeros of $p_{N-1}(x)$, k_{N-1} is the highest coefficient of the polynomial $p_{N-1}(x)$ and $\delta_a(dx)$ is a delta measure on a.

Remark that this theorem means I_N is factorized into N Stieltjes transforms of probability measures. One can regard it as a generalization of the fact that the determinant of a matrix can be factorized into its eigenvalues. This factorization is not unique.

Next we deal with the case that $\mu(dx)$ is supported on the finite interval and absolutely continuous with respect to the Lebesgue measure. Then by using Theorem 1.3, we obtain the corollary.

Corollary 1.4. Let $\mu(dx)$ be supported on $[-1, 1]$ and be absolutely continuous with respect to the Lebesgue measure so that

$$\log \frac{d\mu}{dx} \in L^1(dx) \tag{1.9}$$

Then we have

$$\lim_{N \to \infty} \frac{1}{N} \log I_N(\lambda) = \log \int_{-1}^{1} \frac{1}{\lambda - x} \frac{2}{\pi} \sqrt{1 - x^2}\, dx \tag{1.10}$$

where this convergence is compact uniformly on $\operatorname{Im} \lambda > 0$.

Remark that this corollary implies so-called Wigner's semi-circle law[5] for the distribution of the eigenvalues of Hermitian random matrices and $[-1, 1]$ can be replaced by general $[a, b]$. Various results for more general cases have been obtained by many authors, for example, refs. 4 and 6.

2. A FACTORIZATION OF $I_N(\lambda)$

In this section we will factorize $I_N(\lambda)$ into N products of Stieltjes transforms of probability measures. First we decompose the reciprocal of a polynomial into the partial fraction

$$\prod_{i=1}^{N}\left(\frac{1}{\lambda-x_i}\right)=\sum_{i=1}^{N}\left(\prod_{\substack{l=1\\ l\neq i}}^{N}(x_i-x_l)\right)^{-1}\frac{1}{\lambda-x_i}$$

Then by symmetry of $(x_1, x_2,..., x_N)$ we have

$$\begin{aligned}I_N(\lambda)&=C_N^{-1}\int_{\mathbf{R}^N}\prod_{i=1}^{N}\left(\frac{1}{\lambda-x_i}\right)\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)^2\,\mu_N(dx_1\,dx_2\cdots dx_N)\\
&=\frac{N}{C_N}\int_{\mathbf{R}^N}\frac{1}{\lambda-x_1}\frac{1}{\prod_{j=2}^{N}(x_1-x_j)}\\
&\quad\times\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)^2\,\mu_N(dx_1\,dx_2\cdots dx_N)\\
&=\frac{N}{C_N}\int_{\mathbf{R}^N}\frac{1}{\lambda-x_1}\prod_{2\leqslant k<l\leqslant N}(x_k-x_l)\\
&\quad\times\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)\,\mu_N(dx_1\,dx_2\cdots dx_N)\end{aligned}$$

We prepare a lemma.

Lemma 2.1. Let $f(x_1, x_2,..., x_N)$ be a polynomial of degree less than $\frac{1}{2}N(N-1)$. Then,

$$I_f=\int_{\mathbf{R}^N}f(x_1, x_2,..., x_N)\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)\,\mu_N(dx_1\,dx_2\cdots dx_N)=0 \tag{2.1}$$

Proof. It is sufficient to show the lemma for $f(x_1, x_2,..., x_N)=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_N^{\alpha_N}$ where $\alpha_1+\alpha_2+\cdots+\alpha_N<\frac{1}{2}N(N-1)$. Noting that $\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)$ is the Vandermonde determinant, we obtain

$$I_f = \int_{\mathbf{R}^N} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_N & x_{N-1} & \cdots & x_1 \\ \cdots & \cdots & \cdots & \cdots \\ x_N^{N-1} & x_{N-1}^{N-1} & \cdots & x_1^{N-1} \end{vmatrix} \mu_N(dx_1\, dx_2 \cdots dx_N)$$

$$= \int_{\mathbf{R}^N} \begin{vmatrix} x_N^{\alpha_N} & x_{N-1}^{\alpha_{N-1}} & \cdots & x_1^{\alpha_1} \\ x_N^{\alpha_N+1} & x_{N-1}^{\alpha_{N-1}+1} & \cdots & x_1^{\alpha_1+1} \\ \cdots & \cdots & \cdots & \cdots \\ x_N^{\alpha_N+N-1} & x_{N-1}^{\alpha_{N-1}+N-1} & \cdots & x_1^{\alpha_1+N-1} \end{vmatrix} \mu_N(dx_1\, dx_2 \cdots dx_N)$$

However, when $\alpha_1 + \alpha_2 + \cdots + \alpha_N < \frac{1}{2}N(N-1)$ there exist distinct i and j such that $\alpha_i = \alpha_j$. Then $I_f = 0$. ▮

Corollary 2.2. If $0 \leqslant n < N-1$ then

$$\int_{\mathbf{R}^N} x_1^n \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu_N(dx_1\, dx_2 \cdots dx_N) = 0$$

Proof. It is trivial since the degree of $x_1^n \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l)$ is $\frac{1}{2}(N-1)(N-2) + n$. ▮

Remark 2.3. We define a polynomial of degree $N-1$

$$g_{N-1}(x_1) = \int_{\mathbf{R}^{N-1}} \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu(dx_2) \cdots \mu(dx_N)$$

Corollary 2.2 implies that g_{N-1} is a constant multiple of orthogonal polynomial of degree $N-1$ with respect to the measure $d\mu$, (see ref. 8), where the system of orthogonal polynomials with respect to the measure $d\mu$ is the Schmidt's orthogonalization of $1, x, x^2, \ldots$ with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbf{R}} f(x)\, g(x)\, \mu(dx)$$

Now we consider a linear operator formally defined by

$$(\mathscr{F}_N f)(\lambda) = \int_{\mathbf{R}^N} \frac{f(x_1)}{\lambda - x_1} \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \times \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu_N(dx_1\, dx_2 \cdots dx_N)$$

Then by simple calculation we obtain

$$\lambda(\mathscr{F}_N f)(\lambda) = (\mathscr{F}_N xf)(\lambda) + \int_{\mathbf{R}^N} f(x_1) \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \times \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu_N(dx_1\, dx_2 \cdots dx_N)$$

especially, putting $f(x) = x^m$ we have

$$\lambda(\mathscr{F}_N x^m)(\lambda) = (\mathscr{F}_N x^{m+1})(\lambda) + \int_{\mathbf{R}^N} x_1^m \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \times \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu_N(dx_1\, dx_2 \cdots dx_N)$$

Then we have the following lemma.

Lemma 2.4. Let p be a polynomial of degree less than N. Then

$$\mathscr{F}_N(p) = p(\lambda)\, \mathscr{F}_N(1) \tag{2.2}$$

Proof. It is trivial from Corollary 2.2. ∎

Next we prepare a lemma about the zeros of orthogonal polynomials.

Lemma 2.5. Let $d\mu$ be a measure on $I \subset \mathbf{R}^1$ and be $p_n(x)$ be an orthogonal polynomial of degree n. Then $p_n(x)$ has n distinct zeros in I, that is, there exist n distinct real numbers $\lambda_{n,1}, \lambda_{n,2},..., \lambda_{n,n}$ such that

$$p_n(x) = (\text{const.})(x - \lambda_{n,1})(x - \lambda_{n,2}) \cdots (x - \lambda_{n,n}) \tag{2.3}$$

Proof. See ref. 8. ∎

Remark that since $g_{N-1}(x)$ is an orthogonal polynomial of degree $N-1$, by Lemma 2.5, there exist $\lambda_1, \lambda_2,..., \lambda_{N-1} \in \mathbf{R}$ such that $g_{N-1}(x) = C_{N-1}(x - \lambda_1) \cdots (x - \lambda_{N-1})$.

Now we proceed the calculation of $I_N(\lambda)$. Since $g_{N-1}(x_1)$ is a polynomial of degree $N-1$, using Lemma 2.4, we get

$$I_N(\lambda) = \frac{N}{C_N} \mathscr{F}_N(1) = \frac{N}{C_N} g_{N-1}(\lambda)^{-1} \mathscr{F}_N(g_{N-1})(\lambda)$$

$$= \frac{N}{C_{N-1} C_N} \prod_{i=1}^{N-1} \left(\frac{1}{\lambda - \lambda_i}\right) \mathscr{F}_N(g_{N-1})(\lambda)$$

Since

$$\mathscr{F}_N(f)(\lambda) = \int_{\mathbf{R}^N} \frac{f(x_1)}{\lambda - x_1} \prod_{2 \leqslant k < l \leqslant N} (x_k - x_l) \times \prod_{1 \leqslant i < j \leqslant N} (x_i - x_j)\, \mu_N(dx_1\, dx_2 \cdots dx_N) = \int_{\mathbf{R}} \frac{f(x_1)}{\lambda - x_1} g_{N-1}(x_1)\, \mu(dx_1)$$

we obtain

$$I_N(\lambda) = \frac{N}{C_{N-1} C_N} \prod_{i=1}^{N-1} \left(\frac{1}{\lambda - \lambda_i}\right) \int_{\mathbf{R}} \frac{1}{\lambda - x} g_{N-1}^2(x)\, d\mu(x) = \frac{N}{C_{N-1} C_N} \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left(\frac{1}{\lambda - x_i}\right) \delta_{\lambda_1}(dx_1) \otimes \cdots \otimes \delta_{\lambda_{N-1}}(dx_{N-1}) \otimes g_{N-1}^2(x_N)\, \mu(dx_N)$$

where δ_a is a delta measure on a and $\{\lambda_i\}_{i=1}^{N-1}$ are the $N-1$ zeros of $g_{N-1}(x)$. Then we have the factorization of $I_N(\lambda)$ in the sense above.

Theorem 2.6. Let $p_n(x)$ be an orthonormal polynomial of degree n with respect to the measure $\mu(dx)$. Then we have the following factorization of $I_N(\lambda)$:

$$I_N(\lambda) = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left(\frac{1}{\lambda - x_i}\right) \delta_{\lambda_1}(dx_1) \otimes \cdots \otimes \delta_{\lambda_{N-1}}(dx_{N-1}) \otimes p_{N-1}^2(x_N)\, \mu(dx_N) = \frac{k_{N-1}}{p_{N-1}(\lambda)} \int_{\mathbf{R}^1} \frac{1}{\lambda - x} p_{N-1}(x)^2\, \mu(dx) \tag{2.4}$$

where $\lambda_1,\ldots, \lambda_{N-1}$ are the zeros of $p_{N-1}(x)$, k_{N-1} is the highest coefficient of the polynomial $p_{N-1}(x)$ and $\delta_a(dx)$ is a delta measure on a.

Proof. It is easy to see that $\sqrt{N/C_N C_{N-1}}\, g_{N-1}(x)$ is an orthonormal polynomial $p_n(x)$. ∎

3. THE LIMIT $(1/N) \log I_N(\lambda)/N$ AS $N \to \infty$: A COMPACT SUPPORT CASE

In the previous section we do not put any assumption on the support of a measure $\mu(dx)$. In this section, we assume that the measure $\mu(dx)$ is of compact support and calculate the limit of $(1/N) \log I_N(\lambda)$ as $N \to \infty$.

Let $\mu(dx)$ be a probability measure of compact support $I=[a,b]$. In this case, we can have an estimate: for $\operatorname{Im}\lambda>0$ and any $N\in\mathbf{N}$

$$0<\frac{|\operatorname{Im}\lambda|}{\max(|\lambda-a|^2,|\lambda-b|^2)}\leqslant\left|\int_a^b\frac{1}{\lambda-x}\,p_N^2(x)\,\mu(dx)\right|^2\leqslant\frac{1}{|\operatorname{Im}\lambda|^2}$$

Then we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log\int_a^b\frac{1}{\lambda-x}\,p_N^2(x)\,\mu(dx)=0 \tag{3.1}$$

Hence, by Theorem 2.6, all we have to do is to calculate

$$\lim_{N\to\infty}\frac{1}{N}\log\int_{\mathbf{R}^N}\prod_{i=1}^{N}\left(\frac{1}{\lambda-x_i}\right)\delta_{\lambda_1}(dx_1)\otimes\cdots\otimes\delta_{\lambda_N}(dx_N)$$

Further we assume that $\mu(dx)$ is supported on $[-1,1]$ and absolutely continuous with respect to the Lebesgue measure so that

$$\log\frac{d\mu}{dx}\in L^1(dx) \tag{3.2}$$

In this case Szegö has obtained the asymptotic behavior of $p_N(x)$ and the highest coefficient k_{N0} of $p_N(x)$.

Theorem 3.1 (Szegö). Let f be a function on $[-1,1]$ satisfying with $f\in L^1(dx)$, $\log f\in L^1(dx)$ and $p_n(x)$ be the orthonormal polynomial with respect to $f\,dx$ and k_n is the highest coefficient of $p_n(x)$. Then, as $n\to\infty$, for $\lambda\in\mathbf{C}\backslash[-1,1]$,

$$p_n(\lambda)\simeq C_z z^n \qquad |z|>1, \qquad k_n\simeq D2^n \tag{3.3}$$

where C_z and D are constants and C_z depends only on $z\in\mathbf{C}$, and $\lambda=(z+z^{-1})/2$, $|z|>1$. This holds uniformly for $|z|\geqslant R>1$.

Proof. One can refer to ref. 8, Chapter XII. ∎

Using Szegö's result we immediately obtain the following theorem:

Theorem 3.2. Let $\mu(dx)$ be supported on $[-1,1]$ and be absolutely continuous with respect to the Lebesgue measure so that

$$\log\frac{d\mu}{dx}\in L^1(dx)$$

Then we have

$$\lim_{N \to \infty} \frac{1}{N} \log I_N(\lambda) = \log 2(\lambda - \sqrt{\lambda^2 - 1})$$
$$= \log \int_{-1}^{1} \frac{1}{\lambda - x} \frac{2}{\pi} \sqrt{1 - x^2}\, dx$$

where this convergence is uniform on a compact set in $\text{Im}\,\lambda > 0$.

Proof. By Theorem 2.6, we have

$$\lim_{N \to \infty} \frac{1}{N} \log I_N(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log \frac{1}{(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N)}$$
$$= \lim_{N \to \infty} \frac{1}{N} \log \frac{k_N}{p_N(\lambda)}$$

where k_{N0} is the highest coefficient of the Nth orthogonal polynomial $p_N(\lambda)$. Noting that the conformal mapping $\lambda = (z + z^{-1})/2$ maps the set $\{z \in \mathbf{C};\ |z| > 1\}$ onto $\mathbf{C} \setminus [-1, 1]$ and using Theorem 3.1, we obtain the theorem. ∎

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